**Enrollment No: ______________________Exam Seat No: ______________________**

# C.U.SHAH UNIVERSITY

## Winter Examination-2018

**Subject Name: Functional Analysis**

**Subject Code: 5SC03FUA1** **Branch: M.Sc. (Mathematics)**

**Semester: 3** **Date: 01/12/2018** **Time: 02:30 To 05:30** **Marks: 70**

**Instructions:**

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

---

## SECTION – I

**Q-1** **Attempt the following questions** **(07)**

**a.** Let $Y$ be a closed subspace of a normed space $X$. If $x$ and $y$ are in $X$ then show that $|||x + y + Y||| \leq |||x + Y||| + |||y + Y|||$. **(02)**

**b.** Let $(X, \|\cdot\|)$ be a normed space, and let $f$ be a nonzero linear functional on $X$. Show that $f(E)$ is an open set for every open subset $E$ of $X$. **(02)**

**c.** Show that the closure of a convex set in a normed space is a convex set. **(02)**

**d.** State Jensen's inequality. **(01)**

**Q-2** **Attempt all questions** **(14)**

**a.** Let $\{a_j\}$ and $\{b_j\}$ be sequence in $K$. Let $1 < p < \infty$ and $q \in R$ such that $p + q = pq$. Then show that $\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |b_j|^q\right)^{\frac{1}{q}}$. State the result you use. **(06)**

**b.** Let $l^p = \left\{x = \{x_j\}_{j=1}^{\infty} \; and \; \sum_{j=1}^{\infty} |x_j|^p < \infty\right\}$, consider $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}}$. Then show that $\left(l^p, \|\cdot\|_p\right)$ is a normed space. **(06)**

**c.** In a normed space $X$, show that $|\|x\| - \|y\|| \leq \|x - y\|, \; \forall \, x, y \in X$. **(02)**

**OR**

**Q-2** **Attempt all questions** **(14)**

**a.** Let $X$ be normed space, $Y$ be a closed subspace of $X$ and $Y \neq X$. Let $r$ be a real number such that $0 < r < 1$. Then prove that there exists some $x_r \in X$ such that $\|x_r\| = 1$ and $r < dist\,(x_r, Y) \leq 1$. **(06)**

**b.** Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on a linear space $X$. When is $\|\cdot\|$ stronger than $\|\cdot\|'$ ? Prove that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if there are positive constants $\alpha$ and $\beta$ such that $\alpha\,\|\cdot\| \leq \|\cdot\|' \leq \beta\|\cdot\|$. **(06)**

**c.** Let $X$ and $Y$ be normed spaces. Prove that if $T: X \to Y$ is linear and continuous at 0, then $T$ is bounded on $\overline{U}(0, r)$ for some $r > 0$. **(02)**

**Q-3** **Attempt all questions** **(14)**

**a.** State and prove Hahn-Banach separation theorem. **(07)**

**b.** Let $(X, \|\cdot\|)$ be normed space and $Y$ be a closed subspace of $X$. Show that a sequence $\{x_n + Y\}$ converges to $x + Y$ in $X/Y$ if and only if there is a sequence $\{y_n\}$ in $Y$ such that $\{x_n + y_n\}$ converges to $x$ in $X$. **(04)**

**c.** Let $X$ be a normed space over $K, f \in X'$ and $f \neq 0$. Let $a \in X$ with $f(a) = 1$ and $r > 0$. Then show that $U(a, r) \cap Z(f) = \phi$ if and only if $\|f\| \leq \frac{1}{r}$. **(03)**

**OR**

**Q-3** **Attempt all questions** **(14)**

**a.** Let $X$ be a normed spaced. Show that every continuous functional on any subspace of $X$ has a unique Hahn-Banach extension if and only if $X$ is strictly convex. **(07)**

**b.** Let $X$ and $Y$ be normed spaces. If $X$ is finite dimensional, then show that every linear map from $X$ to $Y$ is continuous. **(04)**

**c.** Let $X$ be a complex normed space. Suppose $f : X \to C$ is linear functional and define $u : X \to R$ by $u(x) = Re\, f(x), x \in X$. Then show that $u$ is real linear functional and $f(x) = u(x) - i\, u(ix), \forall\, x \in X$. **(03)**

## SECTION – II

**Q-4** **Attempt the following questions** **(07)**

**a.** Let $X$ be a normed space, and let $F, G \in BL(X)$. Show that $(FG)^{'} = G'F'$. **(02)**

**b.** If $\{x_n^{'}\}$ is a sequence in $X'$, then prove that $x_n^{'} \xrightarrow{w} x^{'} \Rightarrow x_n^{'} \xrightarrow{w^*} x'$ in $X'$. **(02)**

**c.** Let $P$ be a projection on a normed space $X$. If $P$ is closed, then show that both $Z(P)$ and $R(P)$ are closed in $X$. **(02)**

**d.** State bounded inverse theorem. **(01)**

**Q-5** **Attempt all questions** **(14)**

**a.** Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a fixed $y \in l^q$, define $f_y : l^p \to K$ by $f_y(x) = \sum_{j=1}^{\infty} x(j)\, y(j), x \in l^p$. Then show that $f_y \in (l^p)'$ and $\|f_y\| = \|y\|_q$. **(06)**

**b.** Let $Y$ be a closed subspace of normed space $X$. Show that $X$ is a Banach space if and only if both $Y$ and $X/Y$ are Banach spaces. State the result you use. **(06)**

**c.** Let $F, G \in BL(X, Y)$. Prove that $(F + G)^{'} = F^{'} + G'$. **(02)**

**OR**

**Q-5** **Attempt all questions** **(14)**

**a.** Let $X$ be a separable normed space. Then show that every bounded sequence in $X'$ has a weak* convergent subsequence. **(06)**

**b.** State and prove Uniform boundedness principle. **(06)**

**c.** Let $X$ be a normed space, and let $A \in BL(X)$ be invertible. Show that $A$ is bounded below. **(02)**

**Q-6** **Attempt all questions** **(14)**

**a.** State and prove Closed Graph Theorem. **(07)**

**b.** Let $Z$ be a closed subspace of a normed space $X$. Let $Q: X \to X/Z$ be $Q(x) = x + Z$. Show that $Q$ is continuous and open. State the result you use. **(04)**

**c.** Define weak convergence of a sequence of a normed space. Show that weak limit of a sequence is unique. **(03)**

**OR**

**Q-6** **Attempt all questions** **(14)**

**a.** Let $X$ be a normed space. Define spectrum, eigen spectrum and approximate eigen spectrum of $A \in BL(X)$. If $A$ is of finite rank, then show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$. **(07)**

**b.** Let $X$ and $Y$ be Banach spaces. Show that the product space $X \times Y$, with the norm defined by $\|(x,\ y)\| = \|x\| + \|y\|, (x, y) \in X \times Y$, is Banach space. **(04)**

**c.** Let $X$ and $Y$ be Banach spaces, and $F \in BL(X, Y)$. If $R(F) = Y$, then show that $F^{'}$ is bounded below. **(03)**